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# Local Approximation on Manifolds Using Radial Functions and Polynomials

Jeremy Levesley and David L. Ragozin

**Abstract.** The main focus of this paper is to give error estimates for interpolation on compact homogeneous manifolds, the sphere being an example of such a manifold. The notion of a radial function on the sphere is generalised to that of a spherical kernel on a compact homogeneous manifold. Reproducing kernel Hilbert space techniques are used to generate a pointwise error estimate for spherical kernel interpolation using a positive definite kernel. By exploiting the nice scaling properties of Lagrange polynomials in the tangent space, the error estimate is bounded above by a power of the point separation, recovering, in particular, the convergence rates for radial approximation on spheres.

## §1. Introduction

There is currently significant interest in approximation on spheres, related to many interesting geophysical problems. There are a number of different approximation methods currently available on spheres, including wavelets [3], splines [1], and the subject of this paper, radial functions (sometimes called spherical splines) [3,6]. Error estimates and convergence rates for radial approximation on spheres, of an optimal nature, are recent in vintage [5,4], and rely on some technically demanding mathematics. In this paper we build on an idea of Bos and de Marchi [2] in order to provide convergence rates for radial interpolation on a much wider class of manifolds: the reflexive, compact homogeneous spaces. We will conclude the paper by proving a local spherical harmonic polynomial approximation result on spheres.

Let  $M^d$  be a  $d$ -dimensional compact manifold with a metric  $d(\cdot, \cdot)$  which possesses a transitive group  $G$  of isometries. The group acts transitively in that for every  $x, y \in M^d$ , there exists  $g \in G$  such that  $gx = y$ . If, furthermore, there exists  $g \in G$  such that  $gx = y$  and  $gy = x$ , then  $M^d$  is termed reflexive (for more details see [11]). Such a manifold is a reflexive, compact, metric, homogeneous space. We comment that we can always embed  $M^d$  in

some higher dimensional Euclidean space  $\mathbb{R}^{d+r}$ , the group  $G$  being a compact subgroup of the isometries of  $\mathbb{R}^{d+r}$ . We assume that the metric  $d(\cdot, \cdot)$  on  $M^d$  is inherited from some Euclidean embedding.

We will be interested in interpolation on  $M^d$  using continuous zonal kernels  $k(\cdot, \cdot)$  which have the property that  $k(gx, gy) = k(x, y)$  for all  $x, y \in M^d$  and  $g \in G$ . Such kernels are natural generalisations of radial functions, which are functions only of distance, which is itself  $G$ -invariant. Given a set  $\{x_1, \dots, x_N\} \subset M^d$  and data  $f_1, \dots, f_N \in \mathbb{R}$ , we seek a function of the form

$$s_k(x) = \sum_{i=1}^N \alpha_i k(x, x_i)$$

such that  $s_k(x_i) = f_i$ ,  $i = 1, \dots, N$ .

Given the data  $f_i = f(x_i)$ ,  $i = 1, \dots, N$ , we wish to bound the pointwise error between  $s_k$  and  $f$  at  $x \in M^d$ . We make no assumption on the data points except that they satisfy a point separation criteria in some subset of  $M^d$  (see Section 3).

In Section 2 we will introduce some necessary harmonic analysis on  $M^d$ , discuss the notion of positive definiteness on  $M^d$  in brief, and give a standard error estimate, which we will use in Section 3 to obtain convergence rates. In Section 4 we prove a Whitney type error estimate for local spherical harmonic approximation on the sphere.

## §2. Harmonic Analysis and Error Estimates

For a more complete version of the brief description we give here, see [10,11]. Let  $\Pi_n^{d+r}$  be the degree  $n$  polynomials in  $\mathbb{R}^{d+r}$ , the space in which  $M^d$  is homogeneously embedded. Then, let  $\mathcal{P}_n$  (the spherical polynomials of degree  $n$ ) be the restriction of these polynomials to  $M^d$ . Furthermore, let  $\mathcal{H}_n := \mathcal{P}_n \cap \mathcal{P}_{n-1}^\perp$ , where the orthogonality is with respect to  $d\mu$ , the unique normalised  $G$ -invariant measure on  $M^d$ :

$$[f, g] := \int_{M^d} fg d\mu.$$

Then, we can uniquely decompose  $\mathcal{H}_n$  into irreducible  $G$ -invariant subspaces  $\Xi_{nj}$ , each of dimension  $d_{nj}$ ,  $j = 1, \dots, h_n$ , resulting in the  $G$ -invariant decomposition

$$L_2(M^d) = \bigoplus_{n=0}^{\infty} \bigoplus_{j=1}^{h_n} \Xi_{nj}.$$

Let  $\mathcal{X}_{nj}$  be the orthogonal projection onto  $\Xi_{nj}$ ,  $n = 0, 1, \dots$ , and  $j = 1, \dots, h_n$ , and  $\mathcal{R}_{nj}(\cdot, \cdot)$  be the kernel of this projection. We will consider interpolation using strictly positive definite kernels of the form

$$k(x, y) = \sum_{n=0}^{\infty} \sum_{j=1}^{d_n} a_{nj} \mathcal{R}_{nj}(x, y),$$

where  $a_{nj} > 0$ ,  $n = 0, 1, \dots$ ,  $j = 1, \dots, h_n$ , and

$$\sum_{n=0}^{\infty} \sum_{j=1}^{d_n} d_{nj} a_{nj} < \infty. \quad (1)$$

We will approximate functions from the Hilbert space

$$\mathcal{W} = \{f \in L_2(M^d) : \|f\|^2 := \sum_{n=0}^{\infty} \sum_{j=1}^{d_n} \|\mathcal{X}_{nj} f\|_2^2 / a_{nj} < \infty\},$$

where  $\|\cdot\|_2$  denotes the  $L_2(M^d)$  norm. The associated inner product in  $\mathcal{W}$  is

$$(f, g) := \sum_{n=0}^{\infty} \sum_{j=1}^{d_n} [\mathcal{X}_{nj} f, \mathcal{X}_{nj} g] / a_{nj}.$$

The condition (1) ensures that point evaluation is a continuous linear functional in  $\mathcal{W}$ . It is straightforward to show that  $k$  is the reproducing kernel for the  $\mathcal{W}$ :  $f(x) = (f, k(x, \cdot))$ ,  $f \in \mathcal{W}$ ,  $x \in M^d$ . An immediate consequence of the reproducing kernel property is that  $s_k$  is the interpolant of minimum  $\mathcal{W}$  norm. For, if  $g$  is another interpolant,

$$(s_k - g, s_k) = \sum_{i=1}^N \alpha_i (f - s_k, k(x_i, \cdot)) = \sum_{i=1}^N \alpha_i (f(x_i) - s_k(x_i)) = 0.$$

Therefore,

$$\begin{aligned} (g, g) &= (g - s_k + s_k, g - s_k + s_k) = (g - s_k, g - s_k) + 2(g - s_k, s_k) + (s_k, s_k) \\ &= (g - s_k, g - s_k) + (s_k, s_k), \end{aligned} \quad (2)$$

and the norm minimisation property is established. Now, following the standard arguments, see e.g. [8,9], we have, using the fact that  $s_k$  interpolates  $f$  at  $x_1, \dots, x_N$ ,

$$\begin{aligned} |f(x) - s_k(x)| &= |(f - s_k, k(x, \cdot))| \\ &= |(f - s_k, k(x, \cdot) + \sum_{i=1}^N \beta_i k(x_i, \cdot))| \\ &\leq \|f - s_k\| \|k(x, \cdot) + \sum_{i=1}^N \beta_i k(x_i, \cdot)\| \\ &\leq \|f\| \|k(x, \cdot) + \sum_{i=1}^N \beta_i k(x_i, \cdot)\| \end{aligned}$$

for arbitrary  $\beta_i$ ,  $i = 1, \dots, N$ , where we have used (2) in the final step. Our final error estimate follows from the fact that

$$\begin{aligned} \|k(x, \cdot) + \sum_{i=1}^N \beta_i k(x_i, \cdot)\| &= (k(x, \cdot) + \sum_{i=1}^N \beta_i k(x_i, \cdot), k(x, \cdot) + \sum_{i=1}^N \beta_i k(x_i, \cdot))^{\frac{1}{2}} \\ &= (k(x, x) - 2 \sum_{i=1}^N \beta_i k(x, x_i) + \sum_{i,j=1}^N \beta_i \beta_j k(x_i, x_j))^{\frac{1}{2}}. \end{aligned}$$

Defining

$$P(x, x_1, \dots, x_N) := \inf_{\beta_1, \dots, \beta_N \in \mathbb{R}} (k(x, x) - 2 \sum_{i=1}^N \beta_i k(x, x_i) + \sum_{i,j=1}^N \beta_i \beta_j k(x_i, x_j))^{\frac{1}{2}},$$

we sum these results up in

**Theorem 1.** *Let  $s_k$  be the  $k$ -spline interpolant at  $x_1, \dots, x_N \in M^d$  to  $f \in \mathcal{W}$ . Then, for every  $x \in M^d$ ,*

$$|f(x) - s_k(x)| \leq \|f\| P(x, x_1, \dots, x_N).$$

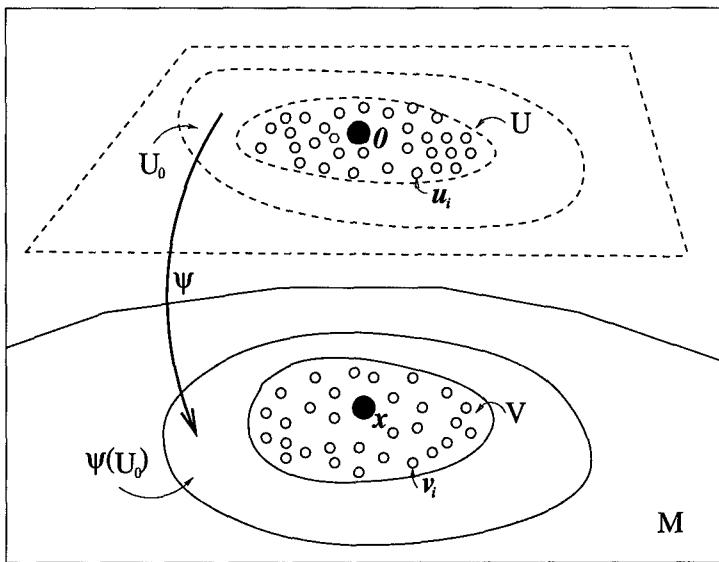
### §3. Convergence Rates for Radial Kernels

In this section we shall give pointwise error estimates in terms of the point separation

$$\rho := \max_{y \in V} \min_{i=1, \dots, N} d(y, x_i),$$

where  $V \subset M^d$  contains  $x$ , the point at which we are measuring the error. As we shall see later in this section, producing a pointwise convergence rate from the error estimate of Theorem 1 requires us to bound Lagrange polynomials related to a subset of the interpolation points. Efforts to produce convergence rates on the sphere foundered because it is difficult to bound the Lagrange polynomials for spherical harmonic interpolation as the interpolation set, with a fixed number of points, scale towards  $x$ . The early error estimates of [3], of  $\mathcal{O}(\rho)$ , were the best known until recently, and only required bounding of the constant Lagrange polynomial for a single point. Light and v. Golitschek [4] proved boundedness for all polynomials on  $S^d$ ,  $d \geq 2$ , and consequently achieved  $\mathcal{O}(\rho^r)$  approximation for radial kernels with  $2r$  continuous derivatives on the sphere.

A very simple proof of the result of Light and v. Golitschek was given by Bos and de Marchi in [2]. What we will do is introduce an analytic coordinate transformation, and construct Lagrange polynomials in the tangent space, which is a  $d$ -dimensional Euclidean space. We will quote a result which uses scaling arguments in Euclidean space which are easy to perform, observing that distance on the manifold and in the tangent are comparable.

Fig. 1. Coordinate chart at  $x$ .

Let  $x \in V \subset M$ . We shall assume the existence of a  $C^\infty$ -chart  $(U_0, \psi)$  with an open subset  $U \subset U_0$  satisfying the following (see Figure 1):

- 1)  $\psi(U) = V$  with  $\psi(0) = x$ ,
- 2)  $U_0 = \{y - z : y, z \in U\}$ .

These conditions ensure the validity of Taylor series arguments which follow. Also, since  $U$  is precompact,  $\psi$  is bi-differentiable and the metric  $d$  is assumed boundedly equivalent to the Euclidean distance on  $\mathbb{R}^{d+r}$ ,

$$c_1 \|y - z\| \leq d(\psi(y), \psi(z)) \leq c_2 \|y - z\|, \quad y, z \in U.$$

Let  $v_1, \dots, v_Q = V \cap X$  be the interpolation points in  $V$ , and redefine  $\rho := \sup_{v \in V} \min_{i=1, \dots, Q} d(v, v_Q)$ . Let  $u_i = \psi^{-1}(v_i)$ ,  $i = 1, \dots, Q$ . Then, from the previous equation we have

$$\eta := \sup_{u \in U} \min_{i=1, \dots, Q} \|u - u_i\| \leq \rho/c_1. \quad (3)$$

It is shown in [7] that provided  $\rho$  and hence  $\eta$  are sufficiently small to guarantee that  $Q > t$ , we can make a selection of interpolation points  $v_1, \dots, v_t$  (assuming a convenient ordering of the points), where  $t = \dim(\Pi_{2r-1}^d)$ , such that the Lagrange polynomials  $p_1, \dots, p_t$  for  $u_1, \dots, u_t$  are bounded at the origin:

$$p_i(0) \leq C_L, \quad i = 1, \dots, t, \quad (4)$$

where  $C_L$  is independent of  $i$  and  $\rho$ . Furthermore,  $u_1, \dots, u_t$  are all contained in  $\eta B_b := \{\eta w : \|w\| \leq b\}$ . We are now ready to prove the main theorem of this paper:

**Theorem 2.** Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  and  $k(\cdot, \cdot) = \phi(d(\cdot, \cdot))$  be strictly positive definite, and  $2r$ -times continuously differentiable in each variable. Let  $x \in S \subset M$ . Suppose that the interpolation points  $x_1, \dots, x_n$  satisfy

$$\sup_{y \in S} \min_{i=1, \dots, n} d(y, x_i) = \rho.$$

Then for all sufficiently small  $\rho$ , if  $s_k$  is the  $k$ -spline interpolant to  $f \in \mathcal{W}$ ,

$$|f(x) - s_k(x)| \leq C \|f\| \rho^r,$$

where  $C$  is independent of  $\rho$ .

**Proof:** First we now choose the coefficients  $\beta_1, \dots, \beta_n$  (appearing in the statement of Theorem 1) as follows. Let  $\beta_i = p_j(0)$  if  $x_i = \psi(u_j)$  for some  $j = 1, \dots, t$ . Set  $\beta_i = 0$  otherwise. This choice of coefficients is made since

$$\sum_{j=1}^t p_j(0) q(u_j) = q(0), \quad (5)$$

for all  $q \in \Pi_{2r-1}^d$ . Then,

$$\begin{aligned} P(x, x_1, \dots, x_N) &\leq \left[ \phi(d(\psi(0), \psi(0))) - 2 \sum_{j=1}^t p_j(0) \phi(d(\psi(0), \psi(u_j))) \right. \\ &\quad \left. + \sum_{i,j=1}^t p_i(0) p_j(0) \phi(d(\psi(u_i), \psi(u_j))) \right]^{1/2}. \end{aligned} \quad (6)$$

Since  $\phi(d(\cdot, \cdot))$  is  $2r$  times continuously differentiable in each variable, for fixed  $w \in U$  we may expand

$$\phi(d(\psi(w), \psi(z))) = \sum_{|\alpha| < 2r} c_\alpha^w (z - w)^\alpha + R_{2r}^w(z), \quad z \in U, \quad (7)$$

where  $R_{2r}^w$  is a Taylor series remainder satisfying

$$R_{2r}^w(z) \leq C_R \|z - w\|^{2r}, \quad (8)$$

for some constant  $C_R$  independent of  $z$  and  $w$ . Putting  $z = w$  in the above expansion we see that

$$c_0^w = \phi(0). \quad (9)$$

Putting (7) into (6) gives

$$\begin{aligned} P(x, x_1, \dots, x_N) &\leq \left[ \phi(0) - 2 \sum_{j=1}^t p_j(0) \left( \sum_{|\alpha| < 2r} c_\alpha^0 (u_j)^\alpha + R_{2r}^0(u_j) \right) \right. \\ &\quad \left. + \sum_{i,j=1}^t p_i(0) p_j(0) \left( \sum_{|\alpha| < 2r} c_\alpha^{u_i} (u_j - u_i)^\alpha + R_{2r}^{u_i}(u_j) \right) \right]^{1/2}. \end{aligned}$$

Using the polynomial reproduction (5), we get

$$\begin{aligned}
 P(x, x_1, \dots, x_N) &\leq \left[ -\phi(0) - 2 \sum_{j=1}^t p_j(0) R_{2r}^0(u_j) \right. \\
 &\quad \left. + \sum_{i=1}^t p_i(0) \left( \sum_{|\alpha| < 2r} c_\alpha^{u_i} (-u_i)^\alpha \right) \right. \\
 &\quad \left. + \sum_{i,j=1}^t p_i(0) p_j(0) R_{2r}^{u_i}(u_j) \right]^{1/2}
 \end{aligned} \tag{10}$$

where we have used (6) and (9) in the above argument.

Now, since the distance function is symmetric, for any  $u, w \in \mathbb{R}^d$ ,

$$\begin{aligned}
 \phi(d(\psi(u), \psi(w))) &= \sum_{|\alpha| < 2r} c_\alpha^w (u - w)^\alpha + R_{2r}^w(u) \\
 &= \sum_{|\alpha| < 2r} c_\alpha^u (w - u)^\alpha + R_{2r}^u(w).
 \end{aligned}$$

In particular, with  $w = 0$  we get

$$\sum_{|\alpha| < 2r} c_\alpha^u (-u)^\alpha = \sum_{|\alpha| < 2r} c_\alpha^0 (u)^\alpha + R_{2r}^0(u) - R_{2r}^u(0).$$

Substituting the last equation into (10) and again using (5) gives

$$\begin{aligned}
 P(x, x_1, \dots, x_N) &\leq \left[ -\phi(0) - 2 \sum_{j=1}^t p_j(0) R_{2r}^0(u_j) \right. \\
 &\quad \left. + \sum_{i=1}^t p_i(0) \left( \sum_{|\alpha| < 2r} c_\alpha^0 (u_i)^\alpha + R_{2r}^0(u) - R_{2r}^u(0) \right) \right. \\
 &\quad \left. + \sum_{i,j=1}^t p_i(0) p_j(0) R_{2r}^{u_i}(u_j) \right]^{1/2} \\
 &\leq \left[ - \sum_{j=1}^t p_j(0) (R_{2r}^0(u_j) + R_{2r}^{u_j}(0)) \right. \\
 &\quad \left. + \sum_{i,j=1}^t p_i(0) p_j(0) R_{2r}^{u_i}(u_j) \right]^{1/2} \\
 &\leq C_R \left( \max_{i,j=1, \dots, t} \{\|u_i - u_j\|, \|u_i\|\} \right)^r \sum_{i,j=1}^t |p_i(0)| |2 + p_j(0)| \\
 &\leq C_R \eta^r \sum_{i,j=1}^t C_L (2 + C_L) \\
 &\leq C \rho^r,
 \end{aligned}$$

using (3), (4) and (8), recalling that  $u_1, \dots, u_t$  are contained in an  $\eta \leq \rho/c_1$  scaled ball of radius  $b$ . Substituting this result in Theorem 1 concludes the proof.  $\square$

#### §4. Whitney-type Estimates

We will now use the coordinate system suggested by Bos and de Marchi [2] to prove Whitney type estimates for approximation using spherical harmonics on  $\mathbb{R}^d$ . This result answers a question posed by L. L. Schumaker during the conference for which these are the proceedings. For  $x = (x_2, \dots, x_{d+1}) \in \mathbb{R}^d$ , define  $\Theta : \mathbb{R}^d \rightarrow S^d$  by  $\Theta(x_2, \dots, x_{d+1}) = ((1 - \sum_{i=2}^{d+1} x_i^2)^{1/2}, x_2, \dots, x_{d+1})$ . This is a smooth parametrisation of a neighbourhood of  $e_1 = (1, 0, \dots, 0) \in S^d$ . As long as  $\sum_{i=2}^{d+1} x_i^2 < \sin^2 \rho$ , then  $d(e_1, \Theta(x)) < \rho$ .

Without loss of generality, we shall consider the approximation of a function  $f \in C^k(S^d)$ , using spherical polynomials, in any spherical neighbourhood  $X_\rho$  of  $e_1$ , where  $\max_{y \in X_\rho} d(e_1, y) = \rho < \pi/2$ . In fact, we will prove

**Theorem 3.** *Let  $f \in C^k(S^d)$ . Then, there exists a degree  $k$  harmonic polynomial  $p_{k-1}$  such that, for every  $\pi/2 > \rho > 0$ ,*

$$\max_{x \in X_\rho} |f(x) - p_{k-1}(x)| \leq C(f)\rho^k,$$

where the constant  $C(f)$  does not depend on  $\rho$ .

**Proof:** The crucial element of this proof is that the coordinate mapping  $\Theta$  maps polynomials of degree  $k$  in  $\mathbb{R}^d$ , the tangent plane at  $e_1$  coordinatised by  $x_2, \dots, x_{n+1}$ , to polynomials of degree  $k$  on the sphere. Since  $f \circ \Theta \in C^k(\Theta^{-1}X_\rho)$ , we can perform the multivariate Taylor series expansion

$$f \circ \Theta(x) = \sum_{|\alpha| < k} c_\alpha x^\alpha + R_k(f, x), \quad (11)$$

where  $R_k(f, x)$  is the remainder satisfying

$$R_k(f, x) \leq C(f)(\max_{y \in X_\rho} |\Theta^{-1}(y)|)^k. \quad (12)$$

Letting  $\theta = \Theta(x)$ , and defining the degree  $k-1$  spherical polynomial

$$p_{k-1}(\theta) := \sum_{|\alpha| < k} c_\alpha (\Theta^{-1}(\theta))^\alpha = \sum_{|\alpha| < r} c_\alpha x^\alpha,$$

equations (11) and (12) tell us that

$$\max_{\theta \in X_\rho} |f(\theta) - p_{k-1}(\theta)| \leq C(f)(\max_{y \in X_\rho} |\Theta^{-1}(y)|)^k.$$

The result follows because  $\max_{y \in X_\rho} |\Theta^{-1}(y)| \leq \rho$ .  $\square$

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